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Stochastic Processes and their Applications 107 (2003) 1–28

stochastic processes and their applications

www.elsevier.com/locate/spa

Asymptotic behavior of the local score of independent and identically distributed random sequences

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Received 14 May 2001; received in revised form 1 April 2003; accepted 4 April 2003

Abstract

Let $(X_n)_{n\geqslant 1}$ be a sequence of real random variables. The local score is $H_n = \max_{1\leqslant i<j\leqslant n}(X_i + \cdots + X_j)$. If $(X_n)_{n\geqslant 1}$ is a "good" Markov chain under its invariant measure, the X_i are centered, we prove that $H_n/\sqrt{n}$ converges in distribution to B_1^* when $n \to +\infty$, where $B_1^* = \max_{0\leqslant u\leqslant 1}|B_u|$ and $(B_u, u \geqslant 0)$ is a standard Brownian motion, $B_0 = 0$. If $(X_n)_{n\geqslant 1}$ a sequence of i.i.d. random variables, $\mathbb{E}(X_1) = \delta/\sqrt{n}$ and $\mathrm{Var}(X_1) = \sigma^2 > 0$, we prove the convergence of $H_n/\sqrt{n}$ to $\sigma\xi_{\delta/\sigma}$ where $\xi_\gamma = \max_{0\leqslant u\leqslant 1}\{(B(u) + \gamma u) - \min_{0\leqslant s\leqslant u}(B(s) + \gamma s)\}$. We approximate the probability distribution function of ξ_γ and we determine the asymptotic behavior of $P(\xi_\gamma \geqslant a)$, $a \to +\infty$.

MSC: 60G17; 60G35; 60J15; 60J20; 60J55; 60J65

Keywords: Brownian motion with drift; Local score

1. Introduction

1.1. Known results

Let $(X_n)_{n\geqslant 1}$ be a sequence of real-valued random variables. Let $S_n = \sum_{k=1}^{n} X_k$, $S_0 = 0$, the associated random walk. Let $H_n = \max_{0\leqslant i<j\leqslant n}(S_j - S_i)$ be the local score

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0304-4149/03/$ - see front matter
doi:10.1016/S0304-4149(03)00061-9

assigned to $(X_n)_{n\geqslant 1}$. The aim of this paper is to study the asymptotic behavior of H_n when $n \to +\infty$, $(X_n)_{n\geqslant 1}$ being either a sequence of i.i.d. random variables or a Markov chain.

The motivations come from biology. The local score is an important tool for DNA sequences analysis. Since the length of DNA is large, the knowledge of the limit behavior of H_n is actually useful.

Some authors have already studied the local score. In a context of queue theory, Iglehart (1972) has investigated the convergence of random variables (i.e. virtual waiting time) which looks like the local score.

When $(X_n)_{n\geqslant 1}$ is a sequence of i.i.d. rv's, Daudin and Mercier (1999) have obtained $\mathbb{P}(H_n < x)$, for any $x > 0$ and $n \geqslant 1$. Let Π be a transition matrix of size x, $P_0 = (1, 0, \ldots, 0)$, and $P_n = (0, 0, \ldots, 1)$. Then $\mathbb{P}(H_n < x) = P_0 \Pi^n P_n'$. In practice, this result is computationally available if n and x are not too large.

When the X_i are i.i.d. rv's with $\mathbb{E}(X_1) < 0$, Dembo and Karlin (1992) have investigated the asymptotic behavior of H_n. More precisely, they proved

$$\lim_{n\to+\infty} \mathbb{P}\left(H_n \leqslant \frac{\ln n}{\lambda} + x\right) = \exp(-K^* \exp(-\lambda x)), \tag{1.1}$$

where K^* and λ depend only on the probability distribution of X_1.

When the X_i are i.i.d. with $\mathbb{E}(X_1) > 0$, the behavior of H_n is drastically different. The strong law of large numbers implies $S_n \underset{n\to+\infty}{\sim} \mathbb{E}(X_1)n$. Obviously, $H_n = \max_{j\leqslant n} Y_j$, where $Y_j = S_j - \min_{i\leqslant j} S_i$. Since $\lim_{n\to+\infty} S_n = +\infty$ a.s., then $-(\min_{j\leqslant n} S_j)$ converges a.s. to a finite r.v., when n goes to infinity. So $Y_j \underset{j\to+\infty}{\sim} S_j$ and $H_n \underset{n\to+\infty}{\sim} \mathbb{E}(X_1)n$. Therefore, $\mathbb{E}(X_1)$ is the parameter which governs a phase transition phenomenon.

1.2. Main results

Here we investigate the case where $(X_i)_{i\geqslant 1}$ is a sequence of r.v's with null or "small" expectation.

We start with the centered case. We suppose that $(X_n)_{n\geqslant 1}$ is either a sequence of centered i.i.d. r.v's with variance $\sigma^2 > 0$ or a "good" Markov centered chain under its invariant probability with parameter σ (see the details in Section 2). In this context, we prove that

$$\frac{H_n}{\sqrt{n}} \underset{n\to+\infty}{\overset{(\mathrm{d})}{\longrightarrow}} \sigma B_1^*, \tag{1.2}$$

where $B_1^* = \max_{0\leqslant u\leqslant 1} |B_u|$, and $(B_u, u \geqslant 0)$ denotes a standard Brownian motion started at 0.

The distribution function of B_1^* is defined as a series (cf. Proposition 2).

Consider a family $\{(X_k^{(N)})_{k\geqslant 1}; N \geqslant 1\}$ of i.i.d. r.v's depending on a parameter N and assume that

$$\lim_{N\to+\infty} \sqrt{N}\,\mathbb{E}(X_1^{(N)}) = \delta \in \mathbb{R}, \qquad \lim_{N\to+\infty} \mathrm{Var}(X_1^{(N)}) = \sigma^2 > 0. \tag{1.3}$$

If the sequence $(X_k)_{k\geqslant 1}$ does not depend on N, then (1.3) implies that $\mathbb{E}(X_1)=0$ and $\mathrm{Var}(X_1)=\sigma^2$. So we obtain the centered case as an instance of the general one.

Eq. (1.3) implies that $\mathbb{E}(X_1^{(N)})\to 0$, when $N\to\infty$.

We prove in Proposition 5 that

$$\frac{H_N^{(N)}}{\sqrt{N}} \underset{n\to\infty}{\overset{(\mathrm{d})}{\to}} \sigma\xi_{\delta/\sigma}, \tag{1.4}$$

where $\xi_\gamma=\max_{0\leqslant u\leqslant 1}\{B(u)+\gamma u-\min_{0\leqslant s\leqslant u}(B(s)+\gamma s)\}$.

Let us summarize the different asymptotic behavior of H_n, n going to infinity.

- If $\mathbb{E}(X_1)<0$, following Dembo and Karlin (1992), the distribution of H_n is approximated by the law of $\ln n/\lambda+\eta$ where η is a r.v. whose c.d.f. is

$$\mathbb{P}(\eta\leqslant x)=\exp(-K^*\exp(-\lambda x)),\quad x\geqslant 0.$$

- If $\mathbb{E}(X_1)>0$, H_n is a.s. equivalent to $\mathbb{E}(X_1)n$.
- If $\mathbb{E}(X_1)=0$, the p.d.f. of H_n can be approximated by the p.d.f. of $(\sigma B_1^*)\sqrt{n}$.
- Suppose that X_1 has a finite variance σ^2 and $\mathbb{E}(X_1)$ is "small" such that $\delta=\sqrt{n}\mathbb{E}(X_1)$. This means that we can find n in a such way that $\sqrt{n}\mathbb{E}(X_1)$ is bounded by a constant. We obtain an approximation of the p.d.f. of H_n by the p.d.f. of $(\sigma\xi_{\delta/\sigma})\sqrt{n}$.

Numerical results about the scope of validity of each approximation is given in Chapter 5 of Etienne (2002).

1.3. More about the p.d.f. of ξ_γ

The distribution function of ξ_γ is difficult to obtain explicitly. We prove that for any fixed $a>0$, $\mathbb{P}(\xi_\gamma>a)$ is the sum of a series (cf. Theorem 9).

Let

$$\xi_\gamma(t)=\max_{0\leqslant u\leqslant t}\left\{B(u)+\gamma u-\min_{0\leqslant s\leqslant u}(B(s)+\gamma s)\right\},\quad t\geqslant 0 \tag{1.5}$$

and

$$T_a=\inf\left\{t\geqslant 0;\ B(t)+\gamma t-\min_{0\leqslant s\leqslant t}(B(s)+\gamma s)>a\right\},\quad a>0. \tag{1.6}$$

Obviously $\xi_\gamma=\xi_\gamma(1)$.

Taylor (1975) and Williams (1976) have determined the Laplace transform of T_a:

$$\mathbb{E}[\mathrm{e}^{-\lambda^2 T_a/2}]=\frac{v\mathrm{e}^{\gamma a}}{v\cosh va+\gamma\sinh va},\quad \lambda>0, \tag{1.7}$$

where $v=\sqrt{\lambda^2+\gamma^2}$.

The distribution of T_a and $(\xi_\gamma(t),t\geqslant 0)$ are linked by the relation

$$\mathbb{P}(T_a<t)=\mathbb{P}(\xi_\gamma(t)>a),\quad \forall t\geqslant 0. \tag{1.8}$$

Suppose that α is a r.v. independent of $(B_t, t \geqslant 0)$ with exponential distribution, then

$$\mathbb{P}(\xi_\gamma(\alpha) > a) = \mathbb{P}(T_a < \alpha) = \mathbb{E}[e^{-T_a}], \quad \forall a > 0. \tag{1.9}$$

Therefore the p.d.f. of $\xi_\gamma(\alpha)$ is explicit:

$$1 - \mathbb{P}(\xi_\gamma(\alpha) \leqslant a) = \frac{v e^{\gamma a}}{v \cosh va + \gamma \sinh va}, \quad \forall a > 0. \tag{1.10}$$

1.4. Tail behavior of ξ_γ

The distribution of ξ_γ is not easy to handle. So we investigate the tail of ξ_γ. We prove (cf. Theorem 4):

$$\mathbb{P}(\xi_\gamma \geqslant a) \underset{a \to \infty}{\sim} 2\sqrt{\frac{2}{\pi}} \frac{1}{a} e^{-(\gamma - a)^2/2}. \tag{1.11}$$

We observe that $a \mapsto \mathbb{P}(\xi_\gamma \geqslant a)$ decreases faster to 0, when $\gamma < 0$. This seems natural since $B_t + \gamma t$ goes to $+\infty$ (resp. $-\infty$) when $\gamma > 0$ (resp. $\gamma < 0$) and $t \to +\infty$.

This remark is connected to the phase transition associated with the sign of $\mathbb{E}(X_1)$.

1.5. Organization of the paper

In Section 2, we study the convergence of H_n when n goes to infinity and $\mathbb{E}(X_1) = 0$. In Section 3, we investigate the asymptotic behavior of H_n when $\mathbb{E}(X_1)$ depends on N. We state the results and detail only short proofs. The more technical proofs are postponed in Section 4.

2. Convergence of the local score in the centered case

Let $(X_n)_{n \geqslant 1}$ be a sequence of real-valued random variables. $(S_k)_{k \geqslant 0}$ denotes the associated random walk:

$$S_0 = 0, \quad S_k = \sum_{i=1}^{k} X_i, \quad k \geqslant 1. \tag{2.1}$$

Let H_n denote the local score

$$H_n = \max_{0 \leqslant i \leqslant j \leqslant n} (S_j - S_i) = \max_{0 \leqslant i \leqslant j \leqslant n} (X_{i+1} + \cdots + X_j). \tag{2.2}$$

We define the sequence of score processes $(H^{(N)})_{N \geqslant 1}$ which are piecewise linear processes:

$$\begin{cases} t \mapsto H^{(N)}(t) \text{ is linear on each interval of the form } \left[\dfrac{j}{N}; \dfrac{j+1}{N}\right], \\ H^{(N)}\left(\dfrac{j}{N}\right) = \dfrac{1}{\sqrt{N}} H_j. \end{cases} \tag{2.3}$$

In this section the sequence $(X_n)_{n\geqslant 1}$ will be either a sequence of i.i.d. centered variables with finite second moment or a stationary and irreducible Markov chain on a finite subset of $\mathbb{R}$. In the first case we set $\sigma^2 = \mathrm{Var}(X_1)$, in the second one we suppose that $\mathbb{E}_v(X_1) = 0$ and

$$\sigma^2 = \mathbb{E}_v(X_1^2) + 2\sum_{k=2}^{\infty} \mathbb{E}_v(X_1 X_k), \tag{2.4}$$

where v is the invariant distribution of $(X_n)_{n\geqslant 0}$.

σ^2 is well defined for series (2.4) is convergent (Billingsley, 1968, p. 166).

We are now able to state the main result of this section:

Theorem 1. *Let $(X_n)_{n\geqslant 1}$ be a sequence of random variables as above.*

Then the sequence of processes $(H^{(N)}(t),\ t \geqslant 0)$ converges in law to the process $(\sigma \max_{0\leqslant u\leqslant s} |B_u|, s \geqslant 0)$, as N tends to infinity.

Proof. We just outline the proof, the complete developments are given in Section 4.1.

Let $B^{(N)}$ be the piecewise linear process defined by

$$B^{(N)}\left(\frac{k}{N}\right) = \frac{1}{\sigma\sqrt{N}} S_k, \quad k \geqslant 0 \tag{2.5}$$

and

$$t \mapsto B^{(N)}(t) \text{ is linear on each interval of the form } [k/N, (k+1)/N]. \tag{2.6}$$

It is well known (Billingsley, 1968) that $(B^{(N)}(s), s \geqslant 0)$ converges to the standard Brownian motion. We easily check that $(H^{(N)}(s), s \geqslant 0)$ may be approached by a continuous function of $(B^{(N)}(s), s \geqslant 0)$ up to a remainder term R_N which converges to 0. According to Paul Levy's theorem (1948, Revuz and Yor, 1991, Chapter II, Theorem 2.3) the process $(B_t - \min_{0\leqslant s\leqslant t} B_s;\ t \geqslant 0)$ has the same law as the process $(|B_t|;\ t \geqslant 0)$. This completes the outline of the proof of Theorem 1. □

An important application of Theorem 1 is the convergence of the local score:

Proposition 2. (1) *$H_n/\sqrt{n}$ converges in distribution, as $n \to \infty$, to σB_1^*, where $B_1^* = \max_{0\leqslant u\leqslant 1}(|B_u|)$.*

(2) *The cumulative distribution function (c.d.f.) of B_1^* is*

$$\mathbb{P}(B_1^* \leqslant x) = \frac{2}{\pi}\sum_{k\in\mathbb{Z}} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8x^2}\right), \quad x \geqslant 0. \tag{2.7}$$

Proof. Theorem 1 implies the convergence in law of the random variable $H_n/\sqrt{n}$:

$$\frac{H_n}{\sqrt{n}} = H^{(n)}\left(\frac{n}{n}\right) = H^{(n)}(1), \quad \forall n \geqslant 0.$$

Equality (2.7) is well known and may be deduced from Borodin and Salminen (1996, p. 146) see 4.6 by a short calculus. □

Remark 3. Theorem 1 implies the convergence of $T_a(H)/a^2$, as a tends to infinity, where $T_a(H) = \inf\{k \geqslant 0;\ H_k > a\},\ a > 0$. Given $a \in \mathbb{R}^+$, then

$$\frac{T_a(H)}{a^2} \underset{a\to\infty}{\overset{(\mathrm{d})}{\longrightarrow}} \frac{1}{\sigma^2\,(B_1^*)^2}. \tag{2.8}$$

Proof. H_k is a non-decreasing process, so

$$\left\{\frac{T_{x\sqrt{N}}(H)}{N} < t\right\} \subset \left\{\frac{H_{[Nt]}}{\sqrt{N}} > x\right\}$$

and

$$\left\{\frac{H_{[Nt]-1}}{\sqrt{N}} > x\right\} \subset \left\{\frac{T_{x\sqrt{N}}(H)}{N} < t\right\}.$$

We know also that $H_{[Nt]-1}/\sqrt{N}$ and $H_{[Nt]}/\sqrt{N}$ have the same limit: $\sigma\sqrt{t}B_1^*$. Then

$$\mathbb{P}\left(\frac{T_{x\sqrt{N}}}{N} < t\right) \underset{N\to\infty}{\longrightarrow} \mathbb{P}(\sigma\sqrt{t}B_1^* > x) = \mathbb{P}\left(\frac{x^2}{\sigma^2\,(B_1^*)^2} < t\right). \tag{2.9}$$

Let $a = x\sqrt{N}$, (2.8) follows immediately. □

3. Convergence in the non-centered case

(1) $(B_t;\ t \geqslant 0)$ denote a standard Brownian motion starting at 0. In this section we suppose that $(X_n)_{n\geqslant 0}$ is a sequence of i.i.d. random variables and that the law of X_1 depends upon N, N being the order of approximation. More precisely, we assume

$$\lim_{N\to\infty} \mathrm{Var}(X_1) = \sigma^2 > 0, \qquad \lim_{N\to\infty} \sqrt{N}\,\mathbb{E}(X_1) = \delta \in \mathbb{R}. \tag{3.1}$$

It is easy to prove (cf. Proposition 5) that $H_N/\sqrt{N}$ converges in distribution, when N goes to infinity, to $\sigma\xi_{\delta/\sigma}$, where

$$\xi_\gamma = \max_{0\leqslant u\leqslant 1}\left\{B(u) + \gamma u - \min_{0\leqslant s\leqslant u}(B(s) + \gamma s)\right\}. \tag{3.2}$$

In the sequel we focus on the law of ξ_γ. It is convenient to introduce

$$\phi^{(\gamma)}(a) = \mathrm{e}^{-\gamma a}\,\mathbb{P}(\xi_\gamma > a), \quad a \geqslant 0. \tag{3.3}$$

Let us briefly detail our approach. We state the main result (Theorem 4) at the end of the subsection.

In Section 2 we have determined the distribution of ξ_γ when $\gamma = 0$. This brings us to remove the drift term, using Girsanov's transformation. Using the pathwise properties of Brownian motion we prove that (cf. Proposition 6 and Theorem 7):

$$\phi^{(\gamma)}(a) = \frac{1}{a}\int_{[0,+\infty[^2} \mathbb{1}_{\{u\leqslant 1\}} \exp\{-\gamma t - \gamma^2 u/2\}\mu_a(u)F_t^{(\gamma)}(1-u,1/a)\,\mathrm{d}u\,\mathrm{d}t, \tag{3.4}$$

where $F_t^{(\gamma)}$ can be expressed as an expectation of a positive r.v.:

$$F_t^{(\gamma)}(x,b) = \mathbb{E}(\mathbb{1}_{\{0\leqslant \tau_t\leqslant x,\ 0\leqslant B^*_{\tau_t}\leqslant 1/b\}}\mathrm{e}^{-\gamma^2\tau_t/2}), \quad x\geqslant 0,\ b\geqslant 0,\ t\geqslant 0. \tag{3.5}$$

The two random variables τ_t and $B^*_{\tau_t}$ are defined as follows:

- τ_t is the first time where the local time at 0 of Brownian motion $(B_u, u\geqslant 0)$ reaches level t,
- $(B^*_t, t\geqslant 0)$ is the process: $B^*_t = \sup_{0\leqslant u\leqslant t}|B_u|$.

For any positive number a, the function μ_a is known (cf. (3.16) and (3.17)).

This allows us to obtain the joint distribution of $(\tau_t, B^*_{\tau_t})$.

The decomposition of the Brownian path up to time τ_t (namely $(B_u; 0\leqslant u\leqslant \tau_t)$), conditionally to $B^*_{\tau_t}$ leads to some recursive structure. This generates two analytic counterparts.

- The density function θ_t of $(\tau_t, B^*_{\tau_t})$ satisfies an integral equation (Proposition 10).
- $F_t^{(\gamma)}$ is solution of an integral equation (cf. (3.20)).

Moreover, relation (3.20) yields to express $F_t^{(\gamma)}$ as sum of a series (Theorem 9). Unfortunately, the coefficients are not explicit and are determined by a recursive algorithm.

However relation (3.20) is rich enough since we determine the decay rate of $a\mapsto P(\xi_\gamma > a),\ a\to\infty$. More precisely

Theorem 4. *For all* γ *in* $\mathbb{R}$:

$$\mathbb{P}(\xi_\gamma\geqslant a) \underset{a\to\infty}{\sim} 2\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-\gamma^2/2}\,\frac{1}{a}\,\mathrm{e}^{\gamma a - a^2/2} = 2\sqrt{\frac{2}{\pi}}\,\frac{1}{a}\,\mathrm{e}^{-(\gamma-a)^2/2}. \tag{3.6}$$

Two proofs of Theorem 4 will be given. The first one is a consequence of Theorem 7 and is postponed in Section 4.4. The second one suggested by the referee will be developed at the end of this section.

(2) We now prove the main results of this section (Theorems 7 and 9).

Only short and easy proofs are given here, the more technical points are postponed in Section 4.

Recall that $(X_n)_{n\geqslant 0}$ will denote a sequence of i.i.d. random variables such that the law of X_1 depends upon a parameter N. We suppose that (3.1) holds. For instance, we can choose

$$\mathbb{P}(X_i = 1) = p_N = \frac{1}{2} + \frac{\delta}{2\sqrt{N}} \quad \text{and} \quad \mathbb{P}(X_i = -1) = q_N = \frac{1}{2} - \frac{\delta}{2\sqrt{N}},$$

for N large enough so that $|\delta/\sqrt{N}| < 1$. Then

$$\mathbb{E}(X_1) = p_N - q_N = \frac{\delta}{\sqrt{N}} \quad \text{and} \quad \mathrm{Var}(X_1) = 1 - \frac{\delta^2}{N}.$$

We set $a_N = \mathbb{E}(X_1)$. Define $B^{(N)}$ as

$$B^{(N)}\left(\frac{k}{N}\right) = \frac{1}{\sigma\sqrt{N}}(S_k - \mathbb{E}(S_k)) = \frac{1}{\sigma\sqrt{N}}(S_k - ka_N), \quad k \geqslant 0 \tag{3.7}$$

and

$$t \mapsto B^{(N)}(t) \text{ is linear on each interval of the form } \left[\frac{k}{N}, \frac{k+1}{N}\right]. \tag{3.8}$$

The process $(H^{(N)}(t), t \geqslant 0)$ is defined by the same procedure as in the centered case, i.e. expression (2.3). It can be shown (Billingsley, 1968, p. 68) that $(B^{(N)}(t), t \geqslant 0)$ converges in distribution to $(B(t), t \geqslant 0)$. $(H^{(N)}(t), t \geqslant 0)$ is a continuous functional of $(B^{(N)}(t), t \geqslant 0)$, this implies the convergence of $H_{[Nt]}/\sqrt{N}$.

Proposition 5. (1) *Let $t > 0$. As N tends to ∞,*

$$\frac{H_{[Nt]}}{\sqrt{N}} = \frac{1}{\sqrt{N}} \max_{1 \leqslant i \leqslant j \leqslant [Nt]} (S_j - S_i) \overset{(\mathrm{d})}{\to} \sigma \xi_{\delta/\sigma}(t),$$

where

$$\xi_\gamma(t) = \max_{0 \leqslant u \leqslant t} \left\{ B(u) + \gamma u - \min_{0 \leqslant s \leqslant u} (B(s) + \gamma s) \right\}. \tag{3.9}$$

(2) *In particular $H_n/\sqrt{n}$ converges in distribution, as $n \to \infty$, to $\sigma\xi_{\delta/\sigma}$, where $\xi_\gamma = \xi_\gamma(1)$.*

Proof. See Section 4.2 for a complete proof. □

Remark. The classical scaling property of Brownian motion (i.e. $(B_s; s \geqslant 0) \overset{(\mathrm{d})}{=} (\sqrt{t}B_{s/t}; s \geqslant 0)$, for any $t > 0$) implies that

$$\xi_\gamma(t) \overset{(\mathrm{d})}{=} \sqrt{t}\,\xi_{\gamma\sqrt{t}}, \quad \text{for any } t > 0. \tag{3.10}$$

This allows us to obtain the distribution of ξ_γ.

Proposition 6. *For all $a > 0$ and $\gamma \in \mathbb{R}$, we set*

$$\phi^{(\gamma)}(a) = \mathrm{e}^{-\gamma a} \mathbb{P}(\xi_\gamma > a). \tag{3.11}$$

Then

$$\phi^{(\gamma)}(a) = \mathbb{E}\left[\mathbb{1}_{\{\tau_Z + T_a < 1\}} \exp\left\{ -\gamma Z - \frac{\gamma^2}{2}(\tau_Z + T_a) \right\} | B^*_{\tau_Z} < a \right], \quad \gamma \in \mathbb{R}, \tag{3.12}$$

where

- τ_t *denotes the first time where the local time at* 0 *of Brownian motion* $(B_t; t \geqslant 0)$ *reaches* t,
- T_a *is the first time where a Bessel process of dimension* 3, *starting at* 0, *hits* a,
- Z *is an exponential random variable of parameter* a (*i.e. its density function is* $(1/a)\mathrm{e}^{-x/a}\mathbb{1}_{\{x>0\}}$).
- $(B^*_u; u \geqslant 0)$ *is the process*: $B^*_u = \sup_{0\leqslant s\leqslant u} |B_s|,\ u \geqslant 0$.
- *for any* $a > 0$, $(B_t;\ t \geqslant 0)$, Z *and* T_a *are independent*.

Proof. We make use, on one hand, of Girsanov's transformation to reduce to the Brownian case and, on second hand, of some sample path properties. See Section 4.3. □

We only need to handle $\phi^{(\gamma)}$. However $\phi^{(\gamma)}$ is equal to $\phi^{(\gamma)}_\gamma$, the function $\phi^{(\gamma)}_\lambda$ being defined as follows:

$$\phi^{(\gamma)}_\lambda(a) = \mathbb{E}\left[\mathbb{1}_{\{\tau_Z+T_a<1\}} \exp\left\{-\gamma Z - \frac{\lambda^2}{2}(\tau_Z + T_a)\right\}\middle| B^*_{\tau_Z} < a\right], \qquad \lambda\in\mathbb{R}. \tag{3.13}$$

In our approach it is not more difficult to deal with $\phi^{(\gamma)}_\lambda$ instead of $\phi^{(\gamma)}$.

Formula (3.12) gives a simple stochastic interpretation of $\phi^{(\gamma)}_\lambda$, but we have to express $\phi^{(\gamma)}_\lambda$ under a more convenient form for computation purpose.

The analytic transcription of (3.12) is the following:

Theorem 7. *Let* $\lambda\in\mathbb{R}$ *be fixed, then for any* $a > 0$

$$\phi^{(\gamma)}_\lambda(a) = \frac{1}{a}\int_{[0,+\infty[^2} \mathbb{1}_{\{u\leqslant 1\}} \exp\{-\gamma t - \lambda^2 u/2\}\mu_a(u) F^{(\lambda)}_t(1-u, 1/a)\,\mathrm{d}u\,\mathrm{d}t, \tag{3.14}$$

where

$$F^{(\lambda)}_t(x,b) = \mathbb{E}(\mathbb{1}_{\{0\leqslant\tau_t\leqslant x,\ 0\leqslant B^*_{\tau_t}\leqslant 1/b\}}\mathrm{e}^{-\lambda^2\tau_t/2}), \qquad x\geqslant 0,\ \ b\geqslant 0,\ \ t\geqslant 0 \tag{3.15}$$

and μ_a *is the density function of* T_a:

$$\mu_a(t) = \frac{1}{a^2}\,\mu_1\left(\frac{t}{a^2}\right) \tag{3.16}$$

and

$$\mu_1(t) = \frac{1}{\sqrt{2\pi}t^{3/2}}\sum_{k\in\mathbb{Z}}\left(-1 + \frac{(1+2k)^2}{t}\right)\exp -\frac{(1+2k)^2}{2t}. \tag{3.17}$$

Furthermore μ_1 *may be expressed as* (Biane et al., 2001, *p.* 8 *and* 24):

$$\mu_1(t) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=-\infty}^{\infty}(-1)^n\mathrm{e}^{-(n^2\pi^2 t)/2}. \tag{3.18}$$

Remark 8. Let $T_a^* = \inf\{t > 0,\ |B_t| > a\}$. $L^0_{T_a^*}$ is an exponential random variable of parameter a.

Since $\{B^*_{\tau_t} < a\} = \{L^0_{T_a^*} > t\}$, obviously $\mathbb{P}(B^*_{\tau_t} < a) = \mathrm{e}^{-t/a}$.

Proof of Theorem 7. The random variables involved in Eq. (3.12) being independent, we have

$$\phi_\lambda^{(\gamma)}(a) = \frac{1}{a}\int_{[0,+\infty[^2} \mathbb{E}[\mathbb{1}_{\{\tau_t+u<1\}}\mathrm{e}^{(-\gamma t-(\lambda^2/2)(\tau_t+u))}|B^*_{\tau_t} < a]\mu_a(u)\mathrm{e}^{-t/a}\,\mathrm{d}u\,\mathrm{d}t, \quad (3.19)$$

where μ_a denotes the density function of T_a.

Using Remark 8, Eq. (3.14) follows immediately. □

We focus our attention on $F_t^{(\lambda)}$. The decomposition of the Brownian path $(B_u, 0 \leqslant u \leqslant \tau_t)$, conditionally to $B^*_{\tau_t}$ leads to some recursive structure. This has an analytic consequence: $F_t^{(\lambda)}$ is solution of an integral equation.

Theorem 9. *Let $\lambda \in \mathbb{R}$ and $t \geqslant 0$ be two fixed parameters.*

(1) *$F^{(\lambda)}$ satisfies the integral equation*

$$F_t^{(\lambda)}(x,a) = F_t^{(\lambda)}(x,0) - t(A^{(\lambda)}F_t^{(\lambda)})(x,a), \quad (x,a)\in\mathbb{R}_+^2, \quad (3.20)$$

with

$$(A^{(\lambda)}\psi)(x,a) = \int_{[0,+\infty[^2} \mathbb{1}_{\{u\leqslant a,\, y\leqslant x\}}\mu^{(2)}_{1/u}(y)\mathrm{e}^{-\lambda^2 y/2}\psi(x-y,u)\,\mathrm{d}y\,\mathrm{d}u, \quad (3.21)$$

$$F_t^{(\lambda)}(x,0) = \frac{t}{\sqrt{2\pi}}\int_0^x \exp\left(-\frac{\lambda^2 z}{2} - \frac{t^2}{2z}\right)\frac{\mathrm{d}z}{z^{3/2}}, \quad (3.22)$$

and $\mu_a^{(2)}(u) = (\mu_a * \mu_a)(u) = (1/a^2)\mu_1^{(2)}(u/a^2)$.

Recall (cf. (Biane et al., 2001)) that

$$\mu_1^{(2)}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{8\sqrt{2}}{\sqrt{\pi}t^{3/2}}\sum_{n=1}^{+\infty} n^2\mathrm{e}^{-2n^2/t}\right). \quad (3.23)$$

(2) *Furthermore $F_t^{(\lambda)}$ can be expressed as a series*

$$F_t^{(\lambda)}(x,a) = \sum_{k=0}^{+\infty}(-1)^k\, t^k\alpha_t^{(k)}(x,a), \quad (3.24)$$

where

$$\alpha_t^{(0)}(x,a) = F_t^{(\lambda)}(x,0), \quad (3.25)$$

$$\alpha_t^{(k+1)}(x,a) = (A^{(\lambda)}\alpha_t^{(k)})(x,a). \quad (3.26)$$

The convergence of (3.24) *holds uniformly for* $(x,a)\in\mathbb{R}_+\times[0,M]$, *for any* $M\geqslant 0$.

(3) *For* $\gamma\in\mathbb{R}$ *and* $a>0$, $\mathbb{P}(\xi_\gamma>a)$ *has the following expansion*:

$$\mathbb{P}(\xi_\gamma>a)=\frac{e^{\gamma a}}{a}\sum_{k\geqslant 0}(-1)^k\int_{[0,+\infty[^2}\mathbb{1}_{\{u\leqslant 1\}}e^{-\gamma t-\lambda^2 u/2}$$
$$\times\mu_a(u)t^k\alpha_t^{(k)}(1-u,1/a)\,du\,dt. \quad (3.27)$$

Proof. See Section 4.5. □

Using Theorem 9 (especially expression (3.20)) we prove that the two-dimensional random variable $(\tau_t,B^*_{\tau_t})$ has for any $t>0$ a density function θ_t which follows an integral equation (3.28). As we notice in 3.1, θ_t is unknown, therefore (3.28) is interesting.

As expression (3.14) shows, $\phi_\lambda^{(\gamma)}$ can be written as an integral of an explicit function of four variables (u,t,x,y) with respect to the positive measure on $\mathbb{R}^4_+:\theta_t(x,y)\,du\,dt\,dx\,dy$. However this expression is not practically useful. In particular, the asymptotic development (3.24) of $F_t^{(\lambda)}$ cannot be deduced from it. This justifies our choice: $F_t^{(\lambda)}$ is the right parameter.

Proposition 10. *Let* $t>0$. *The random variable* $(\tau_t,B^*_{\tau_t})$ *has a density function* θ_t. *Moreover* θ_t *verifies*

$$\theta_t(x,a)=\frac{t}{a^2}\int_{[0,+\infty[^2}\mathbb{1}_{[0,x]\times[0,a]}(y,b)\mu_a^{(2)}(x-y)\theta_t(y,b)\,dy\,db. \quad (3.28)$$

Proof. Let f be the distribution of $(\tau_t,B^*_{\tau_t})$.

Then $f([0,x]\times[0,a])=F^{(0)}(x,1/a)$. We choose $\lambda=0$ and replace a by $1/a$ in Eq. (3.20), we get

$$f([0,x]\times[a,+\infty[)=t\int_0^{1/a}du\left(\int_0^x\mu_{1/u}^{(2)}(y)f([0,x-y]\times[0,1/u])\,dy\right). \quad (3.29)$$

Let η_v be the positive measure $\eta_v(dy)=\mathbb{1}_{\{y>0\}}\tilde{\eta}_v(y)\,dy$, where $\tilde{\eta}_v(y)=\mu_v^{(2)}(y)\mathbb{1}_{\{y>0\}}$.

But

$$(f(.,[0,v])*\eta_v)([0,x])=\int_0^x\mu_v^{(2)}(y)f([0,x-y]\times[0,v])\,dy,$$
$$=\int_0^x\{(f(.,[0,v])*\tilde{\eta}_v)(y)\,dy.$$

The new relation obtained by setting $v=1/a$ in (3.29) implies that $(\tau_t,B^*_{\tau_t})$ has a density θ_t and

$$\theta_t(x,a)=\frac{t}{a^2}\int_0^x\mu_a^{(2)}(x-y)f(dy,[0,a]),$$

$$= \frac{t}{a^2} \int_{[0,+\infty[^2} \mathbb{1}_{[0,x]\times[0,a]}(y,b)\mu_a^{(2)}(x-y)\theta_t(y,b)\,\mathrm{d}b\,\mathrm{d}y. \qquad \square$$

To end up this section, we give a direct proof of Theorem 4 suggested by the referee.

Proof of Theorem 4. Let us introduce some notations. We state

$$X_s = B_s + \gamma s, \quad I_t = \inf_{0 \leqslant s \leqslant t} X_s, \quad Y_t = X_t - I_t, \quad T_a = \inf\{t \geqslant 0:\ Y_t = a\}.$$

Then

$$\mathbb{P}(\xi_\gamma > a) = \mathbb{P}(T_a < 1). \tag{3.30}$$

Williams (1976) and Taylor (1975) have determined the Laplace transform of T_a:

$$\mathbb{E}[\mathrm{e}^{-(\lambda^2/2)T_a}] = \frac{v\mathrm{e}^{\gamma a}}{v\cosh va + \gamma\sinh va}, \tag{3.31}$$

where $v = \sqrt{\lambda^2 + \gamma^2}$.

We are able to invert this expression (cf. step 1 below), i.e. to determine the density function of T_a. Then using (3.30), we obtain the asymptotic behavior of $\mathbb{P}(\xi_\gamma > a),\ a \to \infty$.

(1) By an easy computation we have

$$\mathbb{E}[\mathrm{e}^{-(\lambda^2/2)T_a}] = \frac{2v\mathrm{e}^{(\gamma-v)a}}{\gamma+v}\,\frac{1}{1+[(v-\gamma)/(\gamma+v)]\mathrm{e}^{-2va}},$$

$$= 2v\mathrm{e}^{\gamma a}\left(\sum_{k\geqslant 0}(-1)^k\,\frac{(v-\gamma)^k}{(\gamma+v)^{k+1}}\,\mathrm{e}^{-(2k+1)va}\right).$$

Let L_k be the Laguerre polynomial of order k (Williams, 1976, p. 168). Its Laplace transform is known (Williams, 1976, (7) p. 170):

$$\int_0^{+\infty} \mathrm{e}^{-sx}L_k(x)\,\mathrm{d}x = \frac{(s-1)^k}{s^{k+1}}, \quad s \geqslant 0.$$

This yields to

$$\mathbb{E}[\mathrm{e}^{-(\lambda^2/2)T_a}] = 2\mathrm{e}^{\gamma a}\left(\sum_{k\geqslant 0}(-1)^k\int_0^{+\infty} v\mathrm{e}^{-v(t+(2k+1)a)-\gamma t}L_k(2\gamma t)\,\mathrm{d}t\right). \tag{3.32}$$

Let us recall the integral representation of K_ρ (Watson, 1995, (15), p. 183):

$$K_\rho(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\rho \int_0^{+\infty} \frac{1}{y^{\rho+1}}\,\mathrm{e}^{-(y+z^2/4y)}\,\mathrm{d}y.$$

$K_{1/2}$ and $K_{3/2}$ are known (Watson, 1995, (12), (13) p. 80):

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}, \quad K_{3/2}(z) = \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\left(1+\frac{1}{z}\right).$$

In particular,

$$\rho e^{-\rho(t+(2k+1)a)} = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\lambda^2 x/2} \frac{(t+(2k+1)a)^2 - x}{x^{5/2}} e^{-(\gamma^2 x + (t+(2k+1)a)^2/x)/2} \, dx.$$

Therefore we are able to invert (3.23): T_a has a density ϕ_a and

$$\phi_a(x) = \sqrt{\frac{2}{\pi}} e^{\gamma a} \frac{e^{-\gamma^2 x/2}}{x^{5/2}} \left(\sum_{k \geqslant 0} \psi_{a,k}(x) \right), \tag{3.33}$$

where

$$\psi_{a,k}(x) = (-1)^k \int_0^{+\infty} ((t+(2k+1)a)^2 - x) e^{-\gamma t} L_k(2\gamma t) e^{-(t+(2k+1)a)^2/2x} \, dt. \tag{3.34}$$

(2) We say that $h_a^1(x)$ is uniformly equivalent to $h_a^2(x)$, as $a \to \infty$, x belonging to $[0;1]$, if

$$\lim_{a\to\infty} \left(\sup_{x\in[0;1]} \frac{h_a^1(x)}{h_a^2(x)} \right) = 1.$$

We write $h_a^1(x) \underset{a\to\infty}{\overset{\mathrm{u}}{\sim}} h_a^2(x)$.

We then prove that

$$\psi_{a,0}(x) \underset{a\to\infty}{\overset{\mathrm{u}}{\sim}} xa e^{-a^2/2x}. \tag{3.35}$$

Let $t + a = \sqrt{x}u$ in (3.34) (with $k = 0$):

$$\psi_{a,0}(x) = x^{3/2} e^{\gamma a} \int_{a/\sqrt{x}}^{+\infty} (u^2 - 1) e^{-u^2/2} e^{-\gamma\sqrt{x}u} \, du. \tag{3.36}$$

But

$$(-u e^{-u^2/2})' = (u^2 - 1) e^{-u^2/2}, \tag{3.37}$$

then integrating by part in (3.36) we obtain

$$\psi_{a,0}(x) = x^{3/2} e^{\gamma a} \left(\frac{a}{\sqrt{x}} e^{-a^2/2x} e^{-\gamma a} - \gamma\sqrt{x} \int_{a/\sqrt{x}}^{+\infty} u e^{-u^2/2} e^{-\gamma\sqrt{x}u} \, du \right).$$

Since $u \leqslant (\sqrt{x}/a) u^2$ for any $u \in [a/\sqrt{x}; +\infty[$ and $x \in [0;1]$

$$\frac{xa e^{-a^2/2x}}{1 + \gamma/a} \leqslant \psi_{a,0}(x) \leqslant xa e^{-a^2/2x}.$$

Eq. (3.35) follows immediately.

(3) We claim that

$$\sum_{k\geqslant 0} \psi_{a,k}(x) \underset{a\to\infty}{\overset{\mathrm{u}}{\sim}} \psi_{a,0}(x). \tag{3.38}$$

Suppose $k \geqslant 1$, $a \geqslant 1$ and $x \in [0;1]$.

Then

$$(t+(2k+1)a)^2 \geqslant a^2 \geqslant 1 > x, \quad \forall t \geqslant 0. \tag{3.39}$$

Recall (Widder, 1941, Theorem 17a, p. 168):

$$|L_k(x)| \leqslant e^{x/2}, \quad x \in \mathbb{R}. \tag{3.40}$$

Setting $t+(2k+1)a = \sqrt{x}u$, we obtain

$$|\psi_{a,k}(x)| \leqslant x^{3/2} \int_{(2k+1)a/\sqrt{x}}^{+\infty} (u^2-1)e^{-u^2/2}\,du.$$

By (3.37) the integral can be computed explicitly:

$$|\psi_{a,k}(x)| \leqslant (2k+1)xae^{-(2k+1)^2a^2/2x}.$$

But $x \in]0;1[$, then

$$|\psi_{a,k}(x)| \leqslant (xae^{-a^2/2x})((2k+1)e^{-(2k^2+2k)a^2}).$$

Since $k \geqslant 1$ and $a > 1$,

$$|\psi_{a,k}(x)| \leqslant (xae^{-a^2/2x})((2k+1)e^{-2k^2}e^{-2a^2}). \tag{3.41}$$

This demonstrates (3.38).

(4) Let us end the proof of Theorem 4. Using both (3.30), (3.33), (3.35) and (3.38) we have

$$\mathbb{P}(\xi_\gamma \geqslant a) \underset{a\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\, ae^{\gamma a} I(a),$$

where

$$I(a) = \int_0^1 \frac{1}{x^{3/2}} e^{-\gamma^2 x/2} e^{-a^2/2x}\,dx.$$

Let $x = a^2/(a^2+y)$, we obtain

$$I(a) = \frac{e^{-a^2/2}}{a^2} \int_0^{+\infty} \sqrt{\frac{a^2}{a^2+y}}\, e^{-\gamma^2a^2/2(a^2+y)} e^{-y/2}\,dy.$$

Therefore

$$I(a) \underset{a\to\infty}{\sim} 2\,\frac{e^{-(\gamma^2+a^2)/2}}{a^2}.$$

This proves Theorem 4. □

4. Technical proofs

This section is devoted to the proofs of Theorems 1, 9, 4, Propositions 5, 6 and expressions (2.7), (3.17), (3.18) and (3.23).

4.1. Proof of Theorem 1

Let $(X_n)_{n\geqslant 1}$ be a sequence of r.v.'s and $(S_n)_{n\geqslant 0}$ the random walk:

$$S_0 = 0, \quad S_n = \sum_{k=1}^{n} X_k, \quad n \geqslant 1.$$

We consider two cases:

(a) (X_n) are i.i.d. centered random variables with finite second moment and $\sigma^2 = \mathrm{Var}(X_1)$.

(b) (X_n) is an irreducible Markov chain taking its values in a finite subset of $\mathbb{R}$. We denote by v its invariant distribution. σ^2 is the parameter defined by (2.4).

Given an integer $N \geqslant 0$, we consider the piecewise linear process $B^{(N)}(t)$

$$\begin{cases} B^{(N)}\left(\dfrac{k}{N}\right) = \dfrac{1}{\sigma\sqrt{N}}(S_k - \mathbb{E}(S_k)) = \dfrac{1}{\sigma\sqrt{N}} S_k, \quad k \geqslant 0, \\ t \mapsto B^{(N)}(t) \text{ is linear on each interval of the form } \left[\dfrac{k}{N}, \dfrac{k+1}{N}\right]. \end{cases} \tag{4.1}$$

Our approach is based on the two following results.

Theorem 11 (Billingsley, 1968, pp. 68, 166, 174). *The process $(B^{(N)}(t), t \geqslant 0)$ converges in law, as N tends to $+\infty$, to the standard linear Brownian motion $(B(t), t \geqslant 0)$.*

Theorem 12 (Skorokhod's theorem (Ikeda and Watanabe, 1981, p. 9)). *Let (S, γ) be a complete separable metric space, P and P_n, $n = 1, 2, \ldots$ be probability measures on $(S, \mathscr{B}(S))$ so that $P_n \underset{N\to\infty}{\Rightarrow} P$. Then, we can construct, on a probability space $(\underline{\Omega}, \underline{\mathscr{B}}, \underline{P})$, S-valued random variables X_n, $n = 1, 2, \ldots$ and X such that*

(1) *$P_n = \mathscr{L}(X_n)$, $n = 1, 2, \ldots$ and $P = \mathscr{L}(X)$.*
(2) *X_n converges to X almost everywhere.*

Proof of Theorem 1. The proof is divided into two steps.

Recall that

$$H_k = \max_{0\leqslant i\leqslant j\leqslant k}(S_j - S_i), \quad k \geqslant 0.$$

Let us introduce the linear interpolation of $(H_k)_{k\geqslant 0}$. This function $(H^{(N)}(t); t \geqslant 0)$ depending on the parameter N is defined as follows:

$$H^{(N)}(t) = \frac{1}{\sqrt{N}}\{H_{[Nt]} + (Nt - [Nt])(H_{[Nt]+1} - H_{[Nt]})\}, \quad t \geqslant 0. \tag{4.2}$$

(1) Relation (4.1) implies

$$S_k = \sigma\sqrt{N}\, B^{(N)}\left(\frac{k}{N}\right). \tag{4.3}$$

Then

$$H_{[Nt]} = \sigma\sqrt{N} \max_{0\leqslant i\leqslant j\leqslant [Nt]} \left\{ B^{(N)}\left(\frac{j}{N}\right) - B^{(N)}\left(\frac{i}{N}\right)\right\}$$
$$= \sigma\sqrt{N} \max_{0\leqslant i/N\leqslant j/N\leqslant [Nt]/N} \left\{ B^{(N)}\left(\frac{j}{N}\right) - B^{(N)}\left(\frac{i}{N}\right)\right\}.$$

$B^{(N)}$ being piecewise linear, then the maximum on $\{0 \leqslant i/N \leqslant j/N \leqslant [Nt]/N\}$ is equal to the maximum on $\{0 \leqslant u \leqslant v \leqslant [Nt]/N\}$ and

$$H_{[Nt]} = \sigma\sqrt{N} \max_{0\leqslant u\leqslant v\leqslant [Nt]/N} \{B^{(N)}(v) - B^{(N)}(u)\}.$$

Finally $H^{(N)}(t)$ can be written as follows:

$$H^{(N)}(t) = \sigma\left(\max_{0\leqslant u\leqslant v\leqslant [Nt]/N} \{B^{(N)}(v) - B^{(N)}(u)\} + R_N(t)\right), \tag{4.4}$$

$$R_N(t) = (Nt-[Nt])\left(\max_{0\leqslant u\leqslant v\leqslant ([Nt]+1)/N} \{B^{(N)}(v) - B^{(N)}(u)\}\right.$$
$$\left. - \max_{0\leqslant u\leqslant v\leqslant [Nt]/N} \{B^{(N)}(v) - B^{(N)}(u)\}\right). \tag{4.5}$$

(2) We apply Theorems 11 and 12 with $S=\mathscr{C}([0,T],\mathbb{R})$ and γ the Wiener measure, T being fixed. Then there exist $(\underline{\Omega},\underline{\mathscr{B}},\underline{P})$, $\underline{B}^{(N)}$ and $\underline{B}$ such that $\underline{B}^{(N)}$ converges almost surely to $\underline{B}$ a standard Brownian motion on $(\underline{\Omega},\underline{\mathscr{B}},\underline{P})$.

Let $\underline{R}_N$ (resp. $\underline{H}^{(N)}$) be the process defined by (4.5) (resp. (4.4)) where $B^{(N)}$ is replaced by $\underline{B}^{(N)}$.

But $B^{(N)}$ and $\underline{B}^{(N)}$ have the same law, then

$$(H^{(N)}(t), t\geqslant 0) \overset{(\mathrm{d})}{=} (\underline{H}^{(N)}(t), t\geqslant 0).$$

If we prove that $\underline{H}^{(N)}$ converge a.s., then the previous identity implies the convergence in distribution of $H^{(N)}$.

Since the convergence of $\underline{B}^{(N)}$ holds in the space of continuous functions, for any $t\in[0,T]$:

$$\max_{0\leqslant u\leqslant v\leqslant ([Nt]+1)/N} \{\underline{B}^{(N)}(v) - \underline{B}^{(N)}(u)\} \underset{N\to\infty}{\overset{\text{a.s}}{\to}} \max_{0\leqslant u\leqslant v\leqslant t} \{\underline{B}(v) - \underline{B}(u)\},$$

$$\max_{0\leqslant u\leqslant v\leqslant [Nt]/N} \{\underline{B}^{(N)}(v) - \underline{B}^{(N)}(u)\} \underset{N\to\infty}{\overset{\text{a.s}}{\to}} \max_{0\leqslant u\leqslant v\leqslant t} \{\underline{B}(v) - \underline{B}(u)\}.$$

Moreover as $0\leqslant Nt-[Nt]\leqslant 1$, then

$$\underline{R}_N(t) \underset{N\to\infty}{\overset{\text{a.s.}}{\to}} 0 \quad \text{uniformly in } t\in[0,T].$$

Hence

$$(\underline{H}^{(N)}(t), 0 \leqslant t \leqslant T) \underset{N\to\infty}{\overset{\text{a.s}}{\to}} \left(\sigma \max_{0\leqslant u\leqslant v\leqslant t} \{\underline{B}(v) - \underline{B}(u)\};\ 0 \leqslant t \leqslant T\right). \tag{4.6}$$

We denote $\xi(t) = \max_{0\leqslant u\leqslant v\leqslant t} \{B(v) - B(u)\} = \max_{0\leqslant v\leqslant t} \{B(v) - I(v)\}$ where $I(v) = \min_{0\leqslant u\leqslant v} B(u)$.

Recall that Paul Lévy's theorem (1948, Revuz and Yor, 1991, Chapter II, Theorem 2.3) gives us the following identity:

$$(B(v) - I(v), v \geqslant 0) \overset{(\text{d})}{=} (|B_v|, v \geqslant 0).$$

This ends the proof of Theorem 1. □

4.2. Proof of Proposition 5

This proof is similar to the previous one (see Section 4.1). Let $H^{(N)}$ be the piecewise linear function defined by (4.2). Eq. (4.3) has to be replaced by

$$\frac{1}{\sqrt{N}} S_k = \sigma_N B^{(N)}\left(\frac{k}{N}\right) + \frac{ka_N}{\sqrt{N}} = \sigma_N B^{(N)}\left(\frac{k}{N}\right) + \frac{k}{N}(\sqrt{N}a_N), \tag{4.7}$$

where $a_N = \sqrt{N}\mathbb{E}(X_1)$ and $\sigma_N = \text{Var}(X_1)$.

Suppose $t > 0$. Then

$$\frac{H_{[Nt]}}{\sqrt{N}} = \max_{0\leqslant u\leqslant v\leqslant [Nt]/N} \{\sigma_N B^{(N)}(v) + v(\sqrt{N}a_N) - \sigma_N B^{(N)}(u) - u(\sqrt{N}a_N)\}. \tag{4.8}$$

But $\sqrt{N}a_N$ (resp. σ_N) tends to δ (resp. σ^2), the convergence follows easily.

4.3. Proof of Proposition 6

(1) Let $\phi^{(\gamma)}(a)$ be equal to $\mathrm{e}^{-\gamma a}\mathbb{P}(\xi_\gamma \geqslant a)$. In a first step we establish the following stochastic representation for $\phi^{(\gamma)}$:

$$\phi^{(\gamma)}(a) = \mathbb{E}\left[\mathbb{1}_{\{T_a^* < 1\}} \exp\left(-\gamma L^0_{T_a^*} - \frac{\gamma^2}{2} T_a^*\right)\right]. \tag{4.9}$$

Let f be a Borel bounded function, we have

$$\mathbb{E}[f(\xi^{(\gamma)})] = \mathbb{E}\left[f\left(\max_{0\leqslant u\leqslant 1}\left\{B_u + \gamma u - \min_{0\leqslant s\leqslant u}(B_s + \gamma s)\right\}\right)\right].$$

Let us apply Girsanov's theorem (Revuz and Yor, 1991, Chapter VIII), we get

$$\mathbb{E}[f(\xi^{(\gamma)})] = \mathbb{E}\left[f\left(\max_{0\leqslant u\leqslant 1}\left(B_u - \min_{0\leqslant s\leqslant u} B_s\right)\right)\exp\left\{\gamma B_1 - \frac{\gamma^2}{2}\right\}\right]. \tag{4.10}$$

But Levy's theorem (Revuz and Yor, 1991, Chapter II) gives

$$\left(B_t - \min_{0\leqslant s\leqslant t} B_s, -\min_{0\leqslant s\leqslant t} B_s; t \geqslant 0\right) \overset{(\text{d})}{=} (|B_t|, L_t^0; t \geqslant 0).$$

Then

$$\mathbb{E}[f(\xi^{(\gamma)})] = \mathbb{E}\left[f\left(\max_{0\leqslant s\leqslant 1}|B_s|\right)\exp\left\{\gamma(|B_1| - L_1^0) - \frac{\gamma^2}{2}\right\}\right]. \tag{4.11}$$

Let $(M_t, t \geqslant 0)$ be the process,

$$M_t = \exp\left\{\gamma(|B_t| - L_t^0) - \frac{\gamma^2}{2}t\right\}, \quad t \geqslant 0.$$

M is an exponential martingale since $(|B_t| - L_t^0;\ t \geqslant 0)$ is a Brownian motion.

We restrict ourself to $f = 1_{]a,+\infty[}$, Eq. (4.11) reduces to

$$\mathbb{P}(\xi_\gamma > a) = \mathbb{E}\left[\mathbb{1}_{\{B_1^* > a\}}\exp\left\{\gamma(|B_1| - L_1^0) - \frac{\gamma^2}{2}\right\}\right] = \mathbb{E}[1_{\{B_1^* > a\}}M_1].$$

We have $\{B_1^* > a\} = \{T_a^* < 1\}$ (recall that $B_1^* = \max_{u\leqslant 1}|B_u|$ and $T_a^* = \inf\{t \geqslant 0, |B_t| > a\}$).

Let us introduce $U = T_a^* \wedge 1$. U is a bounded stopping time and $\{T_a^* < 1\} = \{U < 1\}$. Then $\{T_a^* < 1\} \in \mathscr{F}_U$, so that we may apply the stopping time theorem:

$$\begin{aligned}\mathbb{P}(\xi_\gamma > a) &= \mathbb{E}[\mathbb{1}_{\{T_a^* < 1\}}M_1] = \mathbb{E}[\mathbb{1}_{\{T_a^* < 1\}}M_U] \\ &= \mathbb{E}\left[\mathbb{1}_{\{T_a^* < 1\}}\exp\left\{\gamma(a - L_{T_a^*}^0) - \frac{\gamma^2}{2}T_a^*\right\}\right].\end{aligned}$$

This shows (4.9).

(2) We are now able to prove (3.12). The proof is based on decomposition of Brownian path (Vallois, 1991b, Proposition 4). Let us recall this decomposition:

For $a > 0$. Define

$$g = \sup\{t \leqslant T_a^*,\ B_t = 0\}.$$

Then

(i) $T_a^* - g$ and $(B_u, 0 \leqslant u \leqslant \gamma)$ are independent,
(ii) $T_a^* - g \overset{(d)}{=} T_a$,
(iii) conditionally to $L_{T_a^*}^0 = t$, $(B_u, 0 \leqslant u \leqslant g)$ is distributed as $(B_u, 0 \leqslant u \leqslant \tau_t)$ conditioned by $\{B_{\tau_t}^* < a\}$.

We decompose T_a^* as the sum of g and $T_a^* - \gamma$, (3.12) is a straightforward consequence of (4.9). □

4.4. Second proof of Theorem 4

For simplicity $F_y^{(\lambda)}$ will be noted F_y in this section.

Let us start with a preliminary result.

Lemma 13. *Let ψ be the function,*

$$\psi(v) = \int_{\mathbb{R}_+} e^{-\gamma y} F_y\left(\frac{v}{1+v}, 0\right) dy, \quad v > 0. \tag{4.12}$$

Then

$$\psi(v) = \frac{2}{\sqrt{2\pi}} \sqrt{\frac{v}{v+1}} + \psi_1(v), \quad |\psi_1(v)| \leqslant C \frac{v}{1+v}.$$

Proof. Since $F_t(x,0) = \mathbb{E}(\mathbb{1}_{\{0 \leqslant \tau_t \leqslant x,\}} e^{-\lambda^2 \tau_t/2})$ and the density of τ_t is well known (see for example Borodin and Salminen, 1996),

$$\mathbb{P}(\tau_t \in dz) = \frac{t}{\sqrt{2\pi z^3}} \exp\left(-\frac{t^2}{2z}\right) \mathbb{1}_{\{z \geqslant 0\}} dz.$$

Then $F_t(x,0)$ may be written as

$$F_t(x,0) = \frac{t}{\sqrt{2\pi}} \int_0^x \exp\left(-\frac{\lambda^2 z}{2} - \frac{t^2}{2z}\right) \frac{dz}{z^{3/2}}.$$

Therefore,

$$\psi(v) = \frac{1}{\sqrt{2\pi}} \int_0^{v/(1+v)} \frac{1}{z^{1/2}} e^{-\lambda^2 z/2} \int_0^{+\infty} u e^{-(u^2/2 + \gamma u\sqrt{z})} du\, dz. \tag{4.13}$$

We have

$$e^{-x} = 1 + \rho(x),$$

where $|\rho(x)| \leqslant C|x|e^{|x|}$.

In particular $e^{-\gamma u\sqrt{z}} = 1 + \rho(\gamma u\sqrt{z})$, $\psi = \psi_2 + \psi_3$ where

$$\psi_2(v) = \frac{1}{\sqrt{2\pi}} \int_0^{v/(1+v)} \frac{1}{z^{1/2}} e^{-\lambda^2 z/2} \left(\int_0^{+\infty} u e^{-u^2/2} du\right) dz, \tag{4.14}$$

$$\psi_3(v) = \frac{1}{\sqrt{2\pi}} \int_0^{v/(1+v)} \frac{1}{z^{1/2}} e^{-\lambda^2 z/2} \left(\int_0^{+\infty} u e^{-u^2/2} \rho(\gamma u\sqrt{z})\, du\right) dz. \tag{4.15}$$

Clearly

$$\psi_2(v) = \frac{1}{\sqrt{2\pi}} \int_0^{v/(1+v)} \frac{1}{z^{1/2}} e^{-\lambda^2 z/2} dz.$$

But $0 < 1 - e^{-\lambda^2 z/2} < \lambda^2 z/2$ for any $z \geqslant 0$, therefore

$$\psi_2(v) = \frac{2}{\sqrt{2\pi}} \sqrt{\frac{v}{v+1}} + \tilde{\psi}_2(v), \quad |\tilde{\psi}_2(v)| \leqslant C\left(\frac{v}{1+v}\right)^{3/2} \leqslant C\left(\frac{v}{1+v}\right).$$

But

$$|\rho(\gamma u\sqrt{z})| \leqslant C|\delta|u\sqrt{z}e^{|\delta|u\sqrt{z}} \leqslant C|\delta|(ue^{|\delta|u})\sqrt{z}.$$

By the same way

$$|\psi_3(v)| \leqslant C\left(\frac{v}{1+v}\right). \qquad \square$$

Proof of Theorem 4. Let us recall the expression of $\phi_\lambda^{(\gamma)}$ given in Eq. (3.14).

$$\phi_\lambda^{(\gamma)}(a) = \frac{1}{a}\int_{[0,+\infty[^2} \mathbb{1}_{\{u\leqslant 1\}} \exp\{-\gamma y - \lambda^2 u/2\}\, \mu_a(u)\, F_y^{(\lambda)}(1-u, 1/a)\,\mathrm{d}u\,\mathrm{d}y.$$

(1) Let us first prove that $\phi_\lambda^{(\gamma)}(a) \underset{a\to\infty}{\sim} \rho_1(a)$, where

$$\rho_1(a) = \frac{1}{a}\int_{\mathbb{R}_+^2} \mathbb{1}_{\{u\leqslant 1\}} \exp\left\{-\gamma y - \frac{\lambda^2}{2}u\right\} \mu_a(u) F_y^{(\lambda)}(1-u,0)\,\mathrm{d}u\,\mathrm{d}y. \tag{4.16}$$

Recall that

$$F_y^{(\lambda)}(1-u,1/a) = \mathbb{E}[\mathbb{1}_{\{0\leqslant\tau_y\leqslant 1-u, 0\leqslant B^*_{\tau_y}\leqslant a\}}\mathrm{e}^{-\lambda^2\tau_y/2}],$$

so that $\lim_{a\to\infty} F_y^{(\lambda)}(1-u,1/a) = \mathbb{E}[\mathbb{1}_{\{0\leqslant\tau_y\leqslant 1-u\}}\mathrm{e}^{-\lambda^2\tau_y/2}] = F_y^{(\lambda)}(1-u,0)$. Since the convergence is uniform in u, taking the limit over a gives (4.16).

(2) In this step we prove that $\rho_1(a) \underset{a\to\infty}{\sim} \rho_2(a)$, with

$$\rho_2(a) = \frac{2a^2}{\sqrt{2\pi}}\int_{\mathbb{R}_+^2} \mathbb{1}_{\{u\leqslant 1\}}\mathrm{e}^{-\gamma y - \lambda^2 u/2 - a^2/2u}\, F_y^{(\lambda)}(1-u,0)\,\frac{\mathrm{d}u}{u^{5/2}}\,\mathrm{d}y. \tag{4.17}$$

We use the explicit form of μ_a given by Eq. (3.17), the scaling property, and (3.16) then

$$\mu_a(u) = \frac{1}{a^2}\frac{a^3}{\sqrt{2\pi}u^{3/2}}\sum_{k\in\mathbb{Z}}\left(-1 + a^2\frac{(1+2k)^2}{u}\right)\exp\left\{-a^2\frac{(1+2k)^2}{2u}\right\}$$

$$= \frac{a}{\sqrt{2\pi}}\frac{R(u,a)}{u^{3/2}}$$

with

$$R(u,a) = \sum_{k\in\mathbb{Z}}\left(-1 + a^2\frac{(1+2k)^2}{u}\right)\exp\left\{-a^2\frac{(1+2k)^2}{2u}\right\}. \tag{4.18}$$

We split $R(u,a)$ into two parts:

$$R(u,a) = 2\left(\frac{a^2}{u} - 1\right)\mathrm{e}^{-a^2/2u} + \frac{a^2}{u}\mathrm{e}^{-a^2/2u}\left(\sum_{k\in\mathbb{Z}-\{-1,0\}}\beta_k(u,a)\right),$$

$$= \frac{2a^2}{u}\mathrm{e}^{-a^2/2u} + \frac{a^2}{u}\mathrm{e}^{-a^2/2u}\left(-\frac{2u}{a^2} + \sum_{k\in\mathbb{Z}-\{-1,0\}}\beta_k(u,a)\right)$$

with

$$\beta_k(u,a) = \left(-\frac{u}{a^2} + (1+2k)^2\right) \exp\left\{-\frac{a^2}{2u}((1+2k)^2 - 1)\right\}.$$

We prove that the sum, k running over $\mathbb{Z} - \{-1, 0\}$ goes to 0, as $a \to \infty$.

If $a \geqslant 1$ and $u \leqslant 1$, we have

$$\left|-\frac{u}{a^2} + (1+2k)^2\right| \leqslant (1+2k)^2 + 1 \leqslant Ck^2,$$

$$\exp\left\{-\frac{a^2}{2u}\,((1+2k)^2 - 1)\right\} \leqslant \exp\{-2k(k+1)\}.$$

This yields

$$|\beta_k(u,a)| \leqslant Ck^2 \mathrm{e}^{-2k(k+1)}.$$

The dominated convergence theorem implies that

$$\lim_{a\to+\infty} \sum_{k\in\mathbb{Z}-\{-1,0\}} \beta_k(u,a) = 0,$$

uniformly in u.

Furthermore $\lim_{a\to\infty} u/a^2 = 0$ uniformly with respect to $u \in [0,1]$, then

$$R(u,a) \underset{a\to\infty}{\sim} \frac{2a^2}{u}\,\mathrm{e}^{-a^2/2u}.$$

(3) Finally, we check that $\rho_2(a) \underset{a\to\infty}{\sim} 2\sqrt{(2/\pi)}\mathrm{e}^{-\lambda^2/2}\,(1/a)\mathrm{e}^{-a^2/2}$.

We have

$$\rho_2(a) = \frac{2a^2}{\sqrt{2\pi}} \int_0^1 \frac{1}{u^{5/2}} \exp\left\{-\frac{1}{2}\left(\frac{a^2}{u} + \lambda^2 u\right)\right\} \left(\int_{\mathbb{R}_+} \mathrm{e}^{-\gamma y} F_y^{(\lambda)}(1-u, 0)\,\mathrm{d}y\right) \mathrm{d}u.$$

We set $u = 1/(1+v)$, we obtain

$$\rho_2(a) = \frac{2a^2}{\sqrt{2\pi}}\,\mathrm{e}^{-a^2/2} \int_0^{+\infty} \mathrm{e}^{-a^2 v/2} \mathrm{e}^{-\lambda^2/2(1+v)} \sqrt{1+v}\,\psi(v)\,\mathrm{d}v, \tag{4.19}$$

Let us set $u = a^2 v/2$ in Eq. (4.19), then

$$\rho_2(a) = \frac{4}{\sqrt{2\pi}}\,\mathrm{e}^{-a^2/2} \int_0^{+\infty} \mathrm{e}^{-u} \mathrm{e}^{-\lambda^2 a^2/2(a^2+2u)} \sqrt{1 + \frac{2u}{a^2}}\,\psi\left(\frac{2u}{a^2}\right) \mathrm{d}u.$$

Lemma 13 implies that

$$\rho_2(a) = \frac{4}{\sqrt{2\pi}}\,\mathrm{e}^{-a^2/2} \left[\frac{2\sqrt{2}}{\sqrt{2\pi}}\,\frac{1}{a} \int_0^{+\infty} \mathrm{e}^{-u} \mathrm{e}^{-\lambda^2 a^2/2(a^2+2u)} \sqrt{u}\,\mathrm{d}u + \rho_3(a)\right], \tag{4.20}$$

where

$$\rho_3(a) = \int_0^{+\infty} \mathrm{e}^{-\lambda^2 a^2/2(a^2+u^2)} \mathrm{e}^{-u} \sqrt{1 + 2u/a^2}\,\psi_1\left(\frac{2u}{a^2}\right) \mathrm{d}u.$$

The integral on the right-hand side of (4.20) converges as a goes to infinity to

$$e^{-\lambda^2/2}\int_0^{+\infty} e^{-u}\sqrt{u}\,du = e^{-\lambda^2/2}\,\frac{\sqrt{\pi}}{2}.$$

We claim that $|\rho_3(u)|$ is upper bounded by C/a^2, $a \to +\infty$.

Using the upper bound for ψ_1, we obtain

$$|\rho_3(u)| \leqslant C\left(\int_0^{+\infty} u e^{-u}\,du\right)\frac{1}{a^2}.$$

Finally

$$\rho_2(a) \underset{a\to\infty}{\sim} \frac{2\sqrt{2}}{\sqrt{\pi}}\,e^{-\lambda^2/2}\,\frac{1}{a}\,e^{-a^2/2}.$$

As $\mathbb{P}(\xi_\gamma > a) = e^{\gamma a}\phi_\gamma^{(\lambda)}(a)$, we have proved relation (3.6). □

4.5. Proof of Theorem 9

We divide the proof into two steps.

(1) Let $F_t^{(\lambda)}$ be the function defined by (3.15):

$$F_t^{(\lambda)}(x,b) = \mathbb{E}(\mathbb{1}_{\{0\leqslant\tau_t\leqslant x,\ 0\leqslant B^*_{\tau_t}\leqslant 1/b\}}\,e^{-\lambda^2\tau_t/2}), \quad x \geqslant 0,\ b \geqslant 0. \tag{4.21}$$

Here λ and t are fixed. We have

$$F_t^{(\lambda)}(x,b) = \mathbb{E}(\mathbb{1}_{\{0\leqslant\tau_t\leqslant x\}}e^{-\lambda^2\tau_t/2}) - \mathbb{E}(\mathbb{1}_{\{0\leqslant\tau_t\leqslant x,\ B^*_{\tau_t}>1/b\}}e^{-\lambda^2\tau_t/2}). \tag{4.22}$$

Let $B^*_{\tau_t} = u$. Let us define $\gamma = \inf\{s \leqslant \tau_t, |B_s| = u\}$, $g = \sup\{s \leqslant \gamma, B_s = 0\}$, $d = \inf\{s \geqslant \gamma, B_s = 0\}$. Vallois (1991a) proved that conditionally to $B^*_{\tau_t} = u$,

- $g + (\tau_t - d)$, $(\gamma - g)$ et $(d - \gamma)$ are three independent random variables.
- $g + (\tau_t - d)$ is distributed as the first time when the local time of Brownian motion conditioned to stay in $[-u, u]$, reaches t.
- $\gamma - g$ and $d - \gamma$ are distributed as the first time when a Bessel process of dimension 3, started at 0, reaches u. So $(\gamma - g) + (d - \gamma)$ have same law as $T_u + \bar{T}_u$ where $\bar{T}_u$ is an independent copy of T_u.

Since $\tau_t = g + (\tau_t - d) + (\gamma - g) + (d - \gamma)$ and $\mathbb{P}(B^*_{\tau_t} < u) = e^{-t/u}$ (cf. Remark 8), we get

$$\begin{aligned} F_t^{(\lambda)}(x,b) = {} & F_t^{(\lambda)}(x,0) \\ & - t\int_{1/b}^{+\infty}\frac{e^{-t/u}}{u^2}\int_0^{+\infty}\mathbb{E}[1_{\{\tau_t+y<x\}}e^{-(\lambda^2\tau_t)/2}|B^*_{\tau_t}<u] \\ & \times e^{-\lambda^2 y/2}\mu_u^{(2)}(y)\,dy\,du, \end{aligned}$$

$$= F_t^{(\lambda)}(x,0)$$

$$- t\int_{1/b}^{+\infty} \frac{\mathrm{d}u}{u^2}\left(\int_0^{+\infty} \mathbb{E}[1_{\{\tau_t+y<x, B^*_{\tau_t}<u\}}\mathrm{e}^{-\lambda^2\tau_t/2}]\mathrm{e}^{-\lambda^2 y/2}\mu_u^{(2)}(y)\,\mathrm{d}y\right).$$

We set $v = 1/u$, (3.20) follows immediately since we have already established (3.15) in the proof of Lemma 13.

(2) Let K be a positive number and E_K the set of Borel functions ψ defined on $\mathbb{R}_+ \times [0,K]$ such that

$$\sup_{x\geqslant 0, y\leqslant K} |\psi(x,y)| < +\infty.$$

E_K is equipped with the uniform norm.

Let ψ be in E_K, $x \geqslant 0$ and $a \leqslant K$. Then

$$|A^{(\lambda)}\psi\,(x,a)| \leqslant \int_0^a \mathrm{d}u\left(\int_0^x \mu_{1/u}^{(2)}(y)\,\mathrm{e}^{-\lambda^2 y/2}\,|\psi(x-y,u)|\mathrm{d}y\right)$$

$$\leqslant \max_{s\geqslant 0,\,0\leqslant u\leqslant a}|\psi(s,u)|\int_0^a \mathrm{d}u\left(\int_0^x \mu_{1/u}^{(2)}(y)\,\mathrm{d}y\right),$$

$\mu_{1/u}^{(2)}$ being a density function:

$$|A^{(\lambda)}\psi\,(x,a)| \leqslant K \max_{s\geqslant 0,\,0\leqslant u\leqslant K}|\psi(s,u)|.$$

$A^{(\lambda)}$ is thus a continuous linear operator from E_K to E_K.

Clearly $(x,a)\mapsto F_t^{(\lambda)}(x,0)$ belongs to E_K, because

$$0 \leqslant F_t^{(\lambda)}(x,0) \leqslant 1. \tag{4.23}$$

Let us consider the series

$$\Lambda_t(x,a) = \sum_{k=0}^{+\infty}(-1)^k\, t^k \alpha_t^{(k)}(x,a) \tag{4.24}$$

with

$$\alpha_t^{(0)}(x,a) = F_t^{(\lambda)}(x,0)$$

and

$$\alpha_t^{(k+1)}(x,a) = (A^{(\lambda)}\alpha_t^{(k)})(x,a).$$

In order to establish the convergence in E_K, we first prove that

$$\max_{x\geqslant 0, y\leqslant a}|\alpha_t^{(k)}(x,y)| \leqslant \frac{a^k}{k!}\max_{x\geqslant 0, y\leqslant a}|\alpha_t^{(0)}(x,y)| \leqslant \frac{a^k}{k!}. \tag{4.25}$$

We check (4.25) by induction on n.

If $n=0$, obviously (4.25) holds. We suppose that (4.25) is verified for n and we prove that (4.25) is still true, having replaced n by $n+1$.

Let $x \geqslant 0$, $0 \leqslant y \leqslant a$, using again the fact that $\mu^{(2)}_{1/u}$ is a density function, we obtain

$$|A^{(\lambda)}\psi(x,y)| \leqslant \int_{[0,+\infty[^2} \mathbb{1}_{\{u \leqslant y\}} \mu^{(2)}_{1/u}(v) \max_{x\geqslant 0} |\psi(x,u)| \,\mathrm{d}v\,\mathrm{d}u,$$

$$\leqslant \int_0^y \max_{x\geqslant 0} |\psi(x,u)| \,\mathrm{d}u \leqslant \int_0^a \left(\max_{x\geqslant 0,\, u_1\leqslant u} |\psi(x,u_1)| \right) \mathrm{d}u.$$

Therefore

$$\max_{x\geqslant 0, y\leqslant a} |A^{(\lambda)}\psi(x,y)| \leqslant \int_0^a \left(\max_{x\geqslant 0,\, u_1\leqslant u} |\psi(x,u_1)| \right) \mathrm{d}u. \tag{4.26}$$

Therefore (4.25) implies

$$\max_{x\geqslant 0,\, u\leqslant a} |\alpha^{(n+1)}(x,u)| \leqslant \frac{1}{n!}\int_0^a u^n \,\mathrm{d}u = \frac{a^{n+1}}{(n+1)!}.$$

Therefore the series in (4.24) converge in E_K, $A^{(\lambda)}$ is a continuous operator, then

$$F_t^{(\lambda)}(x,a) = \sum_{k=0}^{+\infty} (-1)^k t^k \alpha_t^{(k)}(x,a), \qquad (x,a)\in\mathbb{R}_+^2.$$

(3) Recall the expression of $\phi_\lambda^{(\gamma)}$ in terms of $F_t^{(\lambda)}$.

$$\phi_\lambda^{(\gamma)}(a) = \frac{1}{a}\int_{[0,+\infty[^2} \mathbb{1}_{\{u\leqslant 1\}}\, \mathrm{e}^{-\gamma t-\lambda^2 u/2} F_t^{(\lambda)}(1-u,1/a)\,\mu_a(u)\,\mathrm{d}u\,\mathrm{d}t$$

We are allowed to interchange the sum with respect to k and the double integral if:

$$\sum_{k\geqslant 1}\beta_k < +\infty$$

with

$$\beta_k = \frac{1}{a}\int_{[0,+\infty[^2} \mathbb{1}_{\{u\leqslant 1\}} \mathrm{e}^{-\gamma t-\lambda^2 u/2}\mu_a(u) t^k \alpha_t^{(k)}(1-u,1/a)\,\mathrm{d}u\,\mathrm{d}t.$$

It is well known that $\tau_t \overset{(\mathrm{d})}{=} t^2/B_1^2$, then if $x\leqslant 1$, $t>0$,

$$0 \leqslant \alpha_t^{(0)}(x,a) \leqslant \mathbb{P}(\tau_t \leqslant x) \leqslant \mathbb{P}(\tau_t\leqslant 1) = 2\mathbb{P}(B_1>t) \leqslant \frac{2}{\sqrt{2\pi t}}\,\mathrm{e}^{-t^2/2}.$$

Obviously (4.26) can be modified as follows:

$$\max_{0\leqslant x\leqslant 1, y\leqslant a} |A^{(\lambda)}\psi(x,y)| \leqslant \int_0^a \left(\max_{0\leqslant x\leqslant 1,\, u_1\leqslant u} |\psi(x,u_1)| \right) \mathrm{d}u.$$

Reasoning by induction, we obtain

$$\max_{0\leqslant x\leqslant 1, u\leqslant a} |\alpha_t^{(k)}(x,u)| \leqslant \frac{2}{\sqrt{2\pi t}}\,\mathrm{e}^{-t^2/2}\,\frac{a^k}{k!}.$$

Therefore

$$\beta_k \leqslant \frac{2}{a\sqrt{2\pi}}\int_0^{+\infty} e^{-\gamma t - t^2/2}\left(\frac{t}{a}\right)^k \frac{1}{k!}\,\frac{dt}{\sqrt{t}},$$

$$\sum_{k\geqslant 1}\beta_k \leqslant \frac{2}{a\sqrt{2\pi}}\int_0^{+\infty} e^{-\gamma t - t^2/2}(e^{t/a}-1)\frac{dt}{\sqrt{t}} < +\infty.$$

This implies identity (3.27). □

4.6. *Proof of expression (2.7)*

Recall that $(B_t, t \geqslant 0)$ is a standard Brownian motion, and $B_1^* = \max_{0\leqslant s\leqslant 1}|B_s|$. The cumulative function of B_1^* is known (cf. Borodin and Salminen, 1996, p.146):

$$\mathbb{P}(B_1^* < x) = \frac{1}{\sqrt{2\pi}}\int_{-x}^{x}\sum_{k\in\mathbb{Z}}(e^{-(y+4kx)^2/2} - e^{-(y+2x+4kx)^2/2})\,dy. \tag{4.27}$$

Jacobi's theta function identity (Bellman, 1961) gives us

$$\frac{1}{\sqrt{\pi t}}\sum_{k\in\mathbb{Z}} e^{-(v+k)^2/t} = \sum_{k\in\mathbb{Z}}\cos(2k\pi v)e^{-k^2\pi^2 t}, \qquad v\in\mathbb{R},\ t>0. \tag{4.28}$$

Setting $v = y/4x$ and $t = 1/8x^2$, (4.28) becomes

$$\frac{4x}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} e^{-(4kx+y)^2/2} = \sum_{k\in\mathbb{Z}}\cos(k\pi y/2x)e^{-k^2\pi^2/8x^2}. \tag{4.29}$$

Then

$$\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} e^{-(4kx+y)^2/2} = \frac{1}{4x}\sum_{k\in\mathbb{Z}}\cos(k\pi y/2x)e^{-k^2\pi^2/8x^2}. \tag{4.30}$$

Similarly, setting $v = (y+2x)/4x$ and $t = 1/8x^2$ in (4.28), we obtain

$$\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} e^{-(4kx+2x+y)^2/2} = \frac{1}{4x}\sum_{k\in\mathbb{Z}}(-1)^k\cos(k\pi y/2x)e^{-k^2\pi^2/8x^2}. \tag{4.31}$$

Integrating in y, we obtain the cumulative distribution for B_1^*:

$$\mathbb{P}(B_1^* < x) = \frac{2}{\pi}\sum_{k\in\mathbb{Z}}\frac{(-1)^k}{2k+1}\,e^{-(2k+1)^2\pi^2/8x^2}. \tag{4.32}$$

4.7. *Proof of expression (3.17)*

Let us denote by $(R_x(s), s \geqslant 0)$ a Bessel process of dimension 3 starting at x and $T_a^{(x)}$ the first time where $(R_x(s))_{s\geqslant 0}$ reaches $a(T_a^{(x)} = \inf\{t \geqslant 0; R_x(t) = a\})$.

We claim that $T_a^{(0)}$ admits μ_a as a density function, where

$$\mu_a(t) = \frac{1}{a^2}\mu_1\left(\frac{t}{a^2}\right),$$

$$\mu_1(t) = \frac{1}{\sqrt{2\pi}t^{3/2}}\sum_{k\in\mathbb{Z}}\left(-1+\frac{(1+2k)^2}{t}\right)\exp -\frac{(1+2k)^2}{2t}. \tag{4.33}$$

In (Borodin and Salminen, 1996, p. 339, 2.02) we find the density function of $T_a^{(x)}$, for $0 < x < a$:

$$P(T_a^{(x)} \in \mathrm{d}t) = \frac{a}{x}\Psi_x^{(a)}(t)\mathbb{1}_{\{t\geqslant 0\}}\,\mathrm{d}t = \varphi_x^{(a)}(t)\mathbb{1}_{\{t\geqslant 0\}}\,\mathrm{d}t, \tag{4.34}$$

where

$$\Psi_x^{(a)}(t) = \frac{1}{\sqrt{2\pi t^3}}\sum_{k\in\mathbb{Z}}(a-x+2ka)\exp -\frac{(a-x+2ka)^2}{2t}. \tag{4.35}$$

Let us prove that $\Psi_0^{(a)}(t) = 0$.

For all $t > 0$, we have

$$\begin{aligned}\Psi_0^{(a)}(t) &= \frac{a}{\sqrt{2\pi t^3}}\sum_{k\in\mathbb{Z}}(1+2k)\mathrm{e}^{-(1+2k)^2a^2/2t}\\ &= \frac{a}{\sqrt{2\pi t^3}}\left\{\sum_{k=0}^{+\infty}(1+2k)\mathrm{e}^{-(1+2k)^2a^2/2t}\right.\\ &\quad\left.+\sum_{k=0}^{+\infty}(1+2(-k-1))\mathrm{e}^{-(1+2(-k-1)k)^2a^2/2t}\right\},\\ &= 0.\end{aligned}$$

Then

$$\mu_a(t) = \lim_{x\to 0}\varphi_x^{(a)}(t) = \lim_{x\to 0}\frac{a}{x}(\Psi_x^{(a)}(t) - \Psi_0^{(a)}(t)). \tag{4.36}$$

Differentiating term by term, we obtain

$$\mu_a(t) = \frac{a}{\sqrt{2\pi}t^{3/2}}\sum_{k\in\mathbb{Z}}\left(-1+\frac{a^2(1+2k)^2}{t}\right)\exp -\frac{a^2(1+2k)^2}{2t}. \qquad \square \tag{4.37}$$

4.8. *Proof of (3.18)*

We make use of Poisson expression (Feller, 1966, Chapter XIX, p. 620).

$$\sum_{k\in\mathbb{Z}}\varphi(a+2kb) = \frac{\pi}{b}\sum_{k\in\mathbb{Z}}f\left(\frac{k\pi}{b}\right)\exp\left(\frac{\mathrm{i}k\pi a}{b}\right) \tag{4.38}$$

with

$$\varphi(\alpha) = \int_{\mathbb{R}} e^{i\alpha x} f(x)\, dx.$$

We choose

$$f(x) = \sqrt{\frac{t}{2\pi}} \exp - \frac{t}{2}\left(x - \frac{\pi}{2}\right)^2.$$

f is the density function of $\pi/2 + B_1/\sqrt{t}$, t being a fixed number, then

$$\varphi(\alpha) = e^{i\alpha\pi/2} e^{-\alpha^2/2t}.$$

We set $a = 0$ and $b = 1$ in (4.38), we obtain

$$\sum_{k\in\mathbb{Z}} (-1)^k e^{-2k^2/t} = \sqrt{\frac{\pi t}{2}} \sum_{k\in\mathbb{Z}} \exp - \frac{t}{8}(2k-1)^2\pi^2.$$

We set $t = 4/u\pi^2$:

$$\sum_{k\in\mathbb{Z}} (-1)^k e^{-k^2\pi^2 u/2} = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{u}} \sum_{k\in\mathbb{Z}} \exp - \frac{(2k-1)^2}{2u}. \tag{4.39}$$

Differentiating in respect to u, we obtain (3.18). □

4.9. Proof of expression (3.23)

We keep the notations introduced in the beginning of 4.7.

Let us recall that $\mu_1^{(2)}$ is the density function of $Z = T_1^{(0)} + \tilde{T}_1^{(0)}$, $\tilde{T}_1^{(0)}$ being an independent copy of $T_1^{(0)}$.

The Laplace transform of $T_1^{(0)}$ is well known (Kent, 1978):

$$\mathbb{E}(e^{-\lambda T_1^{(0)}}) = \frac{\sqrt{2\lambda}}{sh\sqrt{2\lambda}}.$$

So that

$$\mathbb{E}(e^{-\lambda Z}) = \left(\frac{\sqrt{2\lambda}}{sh\sqrt{2\lambda}}\right)^2.$$

According to Proposition 1, in Biane et al. (2001, p. 7), this is equivalent to

$$\sqrt{\frac{\pi}{2}} Z \stackrel{(d)}{=} Y,$$

where

$$\mathbb{P}(Y \leqslant y) = \frac{4\pi}{y^3} \sum_{n\geqslant 1} n^2 e^{-\pi n^2/y^2}.$$

A straightforward computation implies (3.23). □

Acknowledgements

We would like to thank a referee for his interesting remarks and suggestions (in particular a direct proof of Theorem 4).

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